8.1

Binomial Theorem for Positive Integral Indices

Learning objectives:

- To study the Binomial Theorem for Positive Integral Indices.
  And
- To practice the related problems.

By actual multiplication

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

These expansions suggest that, when n is a positive integer,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n \text{--(1)}$$

The coefficients in the expansion are denoted by

$$\binom{n}{1} = \frac{n}{1}, \binom{n}{2} = \frac{n(n-1)}{1\cdot 2}, \binom{n}{3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, \ldots$$

The coefficient of any term may be expressed as

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r} = \frac{n!}{(n-r)!r!}$$

The numbers denoted by the symbol $\binom{n}{r}$, where r and n are positive integers with $r \le n$ [read: "${}^nC_r$" or "n choose r"], are called the *binomial coefficients*, since they appear as the coefficients in the expansion of $(a+b)^n$.

The expansion in (1) is known as binomial theorem expansion. Thus, the binomial theorem for positive integral indices is given by

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

$$= x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{2}x^2y^{n-2} + \binom{n}{1}xy^{n-1} + y^n$$

Example 1:

Expand $(3x+2y^2)^5$.

$$(3x+2y^2)^5 = (3x)^5 + \frac{5}{1}(3x)^4(2y^2) + \frac{5\cdot 4}{1\cdot 2}(3x)^3(2y^2)^2 + \frac{5\cdot 4}{1\cdot 2}(3x)^2(2y^2)^3 + \frac{5}{1}(3x)(2y^2)^4 + (2y^2)^5$$

$$= 243x^5 + 810x^4y^2 + 1080x^3y^4 + 720x^2y^6 + 240xy^8 + 32y^{10}$$

Example 2:

How many subsets are there of a set consisting of n elements?

Solution:

There are $\binom{n}{k}$ subsets of size k, $0 \le k \le n$. Therefore

$$\sum_{k=0}^{n}\binom{n}{k}1^k 1^{n-k} = (1+1)^n$$

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

The desired answer is 2^n.

We consider some special cases of the binomial expansion.

$$(x-y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(-y)^k$$

$$= x^n - \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 - \binom{n}{3}x^{n-3}y^3 + \cdots + (-1)^n y^n$$

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n$$

$$(1-x)^n = \sum_{k=0}^{n}\binom{n}{k}(-x)^k$$

$$= 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \cdots + (-1)^n x^n$$

Example 3:

Evaluate $(1.02)^{12}$ correct to four decimal places.

Solution:

$$(1.02)^{12} = (1+0.02)^{12}$$

$$= 1 + 12(0.02) + \frac{12\cdot 11}{2\cdot 1}(0.02)^2 + \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}(0.02)^3 + \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}(0.02)^4 + \cdots$$

$$= 1 + 0.24 + 0.0264 + 0.00176 + 0.0008 + \cdots$$

$$= 1.26824$$

Thus, $(1.02)^{12} = 1.2682$ correct to four decimal places.

IP1:

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6=$$

Solution:

$$\left(1+\sqrt{2}\right)^6=1+{}^6C_1\left(\sqrt{2}\right)+{}^6C_2\left(\sqrt{2}\right)^2+{}^6C_3\left(\sqrt{2}\right)^3$$
$$+{}^6C_4\left(\sqrt{2}\right)^4+{}^6C_5\left(\sqrt{2}\right)^5+{}^6C_6\left(\sqrt{2}\right)^6$$

$$\left(1-\sqrt{2}\right)^6=1-{}^6C_1\left(\sqrt{2}\right)+{}^6C_2\left(\sqrt{2}\right)^2-{}^6C_3\left(\sqrt{2}\right)^3$$
$$+{}^6C_4\left(\sqrt{2}\right)^4-{}^6C_5\left(\sqrt{2}\right)^5+{}^6C_6\left(\sqrt{2}\right)^6$$

$$Now, \left(1+\sqrt{2}\right)^6+\left(\sqrt{2}-1\right)^6=2\left[1+{}^6C_2(2)+{}^6C_4(4)+{}^6C_6(8)\right]$$
$$=2\left[1+30+60+8\right]=198$$

IP2:

For $x \neq 0, \quad \left(x^2 - \frac{3}{x}\right)^4 =$

Solution:

$$\left(x^2 - \frac{3}{x}\right)^4 = {}^4C_0\left(x^2\right)^4 - {}^4C_1\left(x^2\right)^3\left(\frac{3}{x}\right) + {}^4C_2\left(x^2\right)^2\left(\frac{3}{x}\right)^2$$

$$- {}^4C_3\left(x^2\right)\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 - 4x^6.\left(\frac{3}{x}\right) + 6x^4\left(\frac{9}{x^2}\right) - 4x^2\left(\frac{27}{x^3}\right) + \left(\frac{81}{x^4}\right)$$

$$= x^8 - 12x^5 + 54x^2 - \frac{108}{x} + \frac{81}{x^4}$$

IP3:

Which is larger $(1.01)^{1000000}$ or $10,000$.

Solution:

Splitting 1.01 and using binomial theorem to write the first few terms we have

$$
\begin{aligned}
(1.01)^{1000000} &= (1+0.01)^{1000000} \\
&= 1 + {}^{1000000}C_1(0.01) + \textit{positive terms} \\
&= 1 + 1000000 \times 0.01 + \textit{positive terms} \\
&= 1 + 10000 + \textit{positive terms}
\end{aligned}
$$

$$(1.01)^{1000000} = 10001 + \cdots > 10000$$

$$\therefore (1.01)^{1000000} > 10000$$

IP4:

Using Binomial theorem, prove that $6^n - 5n$ always leaves remainder 1 when divided by 25

Solution:

We have $(1+x)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^r$

Put $x = 5$, *we get*

$$(1+5)^n = 6^n = \sum_{r=0}^{n} {}^nC_r \cdot 5^r$$

$$\Rightarrow 6^n = {}^nC_0 + {}^nC_1 5 + {}^nC_2 5^2 + {}^nC_3 5^3 + ... + {}^nC_n 5^n$$

$$\Rightarrow 6^n = 1 + 5n + 5^2\left[{}^nC_2 + {}^nC_3 5 + ... + {}^nC_n 5^{n-2}\right]$$

$$\Rightarrow 6^n - 5n = 1 + 25 \cdot k, \quad where\ k = \left[{}^nC_2 + {}^nC_3 5 + ... + {}^nC_n 5^{n-2}\right]$$

$$\Rightarrow 6^n - 5n = 25 \cdot k + 1$$

This shows that when divided by 25, $6^n - 5n$ leaves remainder 1.

P1.

$$\left(\sqrt{3}+1\right)^{5}-\left(\sqrt{3}-1\right)^{5}=$$

Solution:

$$\left(1+\sqrt{3}\right)^5 = 5_{C_0} + 5_{C_1}\left(\sqrt{3}\right) + 5_{C_2}\left(\sqrt{3}\right)^2 + 5_{C_3}\left(\sqrt{3}\right)^3 + 5_{C_4}\left(\sqrt{3}\right)^4 + 5_{C_5}\left(\sqrt{3}\right)^5$$

$$\left(1-\sqrt{3}\right)^5 = 5_{C_0} - 5_{C_1}\left(\sqrt{3}\right) + 5_{C_2}\left(\sqrt{3}\right)^2 - 5_{C_3}\left(\sqrt{3}\right)^3 + 5_{C_4}\left(\sqrt{3}\right)^4 - 5_{C_5}\left(\sqrt{3}\right)^5$$

$$Now, \left(1+\sqrt{3}\right)^5 + \left(1-\sqrt{3}\right)^5 = 2\left[1 + 5_{C_2}(3) + 5_{C_4}(9)\right]$$

$$\Rightarrow \left(\sqrt{3}+1\right)^5 - \left(\sqrt{3}-1\right)^5 = 2[1+30+45] = 152$$

P2:

The coefficient of x^7 in the expansion $\left(1+x^2\right)^4(1+x)^7$ is

Solution:

$$\left(1+x^2\right)^4 (1+x)^7$$

$$=\left(1+{}^4C_1x^2+{}^4C_2x^4+{}^4C_3x^6+{}^4C_4x^8\right).$$

$$\left(1+{}^7C_1x+{}^7C_2x^2+{}^7C_3x^3+{}^7C_4x^4+{}^7C_5x^5+{}^7C_6x^6+{}^7C_7x^7\right)$$

$$=\left(1+4x^2+6x^4+4x^6+x^8\right)$$

$$\left(1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7\right)$$

Now, the coefficient of x^7 in the expansion is

$$1+4(21)+6(35)+4(7)=323$$

P3.

Find the approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution:

We have

$$(0.99)^5 = (1-0.01)^5$$

$$= \left(1-\frac{1}{100}\right)^5$$

$$= {}^5C_0 - {}^5C_1 \times \frac{1}{100} + {}^5C_2 \times \left(\frac{1}{100}\right)^2 - {}^5C_3 \times \left(\frac{1}{100}\right)^3 + {}^5C_4 \times \left(\frac{1}{100}\right)^4 - {}^5C_5 \times \left(\frac{1}{100}\right)^5$$

$$= 1 - \frac{5}{100} + \frac{10}{(100)^2} - \frac{10}{(100)^3} + \frac{5}{(100)^4} - \frac{1}{(100)^5}$$

$= 1 - 0.05 + 0.0001$ (Neglecting 4^{th} and other terms)

$$(0.99)^5 = 0.951$$

2. Find $(a+b)^4-(a-b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4$.

3. Find $(x+1)^6 + (x-1)^6$. Hence evaluate

$$\left(\sqrt{2}+1\right)^6 - \left(\sqrt{2}-1\right)^6.$$

4. Use the binomial theorem to evaluate $(1.06)^6$ correct to four decimal places.

5. Using binomial theorem, evaluate each of the following:

a. $(96)^3$

b. $(98)^5$

c. $(102)^5$

d. $(101)^4$

6. Using binomial theorem, indicate which is larger $(1.1)^{10000}$ or 1000.

7. Using binomial theorem prove that $50^n - 49n - 1$ is divisible by 49^2 for all positive integers n.

8. Using binomial theorem, prove that $625^n - 48n - 1$ is divisible by 576 for all positive integers n.

9. If n is a positive integer, then prove that $81^n + 20n - 1$ is divisible by 100.

Solution:

$$5^{4n} + 52n - 1$$

$$= 25^{2n} + 52n - 1$$

$$= (26 - 1)^n + 52n - 1$$

$$= {}^{2n}C_0(26)^{2n} - {}^{2n}C_1(26)^{2n-1} + {}^{2n}C_2(26)^{2n-2} - \ldots$$

$$+ {}^{2n}C_{2n-2}(26)^2 - {}^{2n}C_{2n-1}(26) + {}^{2n}C_{2n} + 52n - 1$$

$$= (26)^2\left[(26)^{2n-2} - {}^{2n}C_1(26)^{2n-3} + \ldots + {}^{2n}C_{2n-2}\right] - (2n)26 + 1 + 52n - 1$$

$$= 676\left[(26)^{2n-2} - {}^{2n}C_1(26)^{2n-3} + \ldots + {}^{2n}C_{2n-2}\right]$$

which is divisible by 676

Exercises:

1. Expand the following using binomial theorem.

a. $(2x + 3y)^6$

b. $(4x + 5y)^7$

c. $\left(\frac{2}{3}x + \frac{7}{4}y\right)^5$

d. $\left(\frac{2p}{5} - \frac{3q}{7}\right)^6$

e. $(3 + x - x^2)^4$

f. $\left(x^2 + \frac{1}{2}y\right)^6$

8.2

General and Middle Terms

Learning objectives:

- To find the general and middle terms in a binomial expansion.
  And
- To practice the related problems.

Let n be a positive integer. Let $T_1, T_2, T_3, \ldots$ denote the terms of expansion of a binomial

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
$$= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{r} x^{n-r} y^r +$$
$$\ldots + \binom{n}{2} x^2 y^{n-2} + \binom{n}{1} x y^{n-1} + y^n$$

Then

$$T_1 = \binom{n}{0} x^n y^0$$
$$T_2 = \binom{n}{1} x^{n-1} y^1$$
$$T_3 = \binom{n}{2} x^{n-2} y^2$$

We call the $(r+1)^{\text{th}}$ term of the expansion the ***general term*** of the expansion. It is denoted by $\boldsymbol{T_{r+1}}$.
The ***general term*** of a binomial expansion $(x+y)^n$ is given by

$$\boldsymbol{T_{r+1} = \binom{n}{r} x^{n-r} y^r}$$

Example 1:
Find the fifth term in the expansion of $(2x+3y)^{12}$.
Solution:
The fifth term is given by

$$T_5 = T_{4+1} = \binom{12}{4} (2x)^{12-4} (3y)^4$$
$$= \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1} (2)^8 x^8 (3)^4 y^4$$
$$= 495(256x^8)(81y^4) = 10{,}264{,}320x^8y^4$$

Example 2:
Find the ninth term of $\left(x - \frac{1}{x^{1/2}}\right)^{12}$.
Solution:

$$T_9 = T_{8+1} = \binom{12}{8} x^4 \left(-\frac{1}{x^{1/2}}\right)^8$$
$$= \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1} \cdot x^4 \cdot \frac{1}{x^4} = 495$$

Example 3:
Find the middle term in the expansion of $\left(x^{2/3} + \frac{1}{x^{1/2}}\right)^{10}$.
Solution:
Since there are 11 terms in all, the middle term is the sixth.

$$T_6 = T_{5+1} = \binom{10}{5} \left(x^{2/3}\right)^5 \left(\frac{1}{x^{1/2}}\right)^5$$
$$= 252x^{10/3} \frac{1}{x^{5/2}} = 252x^{5/6}$$

Middle term:
The middle term of a binomial expansion depends upon whether the index n is even or odd.
If $\boldsymbol{n}$ is even, the number of terms in the expansion is $\boldsymbol{n+1}$, which is odd. Therefore, the middle term is given by $\boldsymbol{T_{\text{mid}} = \frac{n+1+1}{2} = \frac{n}{2} + 1}$

In example 3, where $n = 10$, the middle term is $\frac{10}{2} + 1 = 6$.
If $\boldsymbol{n}$ is odd, then $\boldsymbol{n+1}$ is even, so there will be two middle terms in the expansion:

$$\boldsymbol{\frac{n+1}{2}} \quad \textbf{and} \quad \boldsymbol{\frac{n+3}{2}}$$

For example, in the expansion of $(x+y)^5$, the two middle terms are $\frac{5+1}{2} = 3$ and $\frac{5+3}{2} = 4$.

Note:

In expansions of the type $\left(x + \frac{1}{x}\right)^{2n}$,the middle term

$$T_{\text{mid}} = T_{n+1} = \binom{2n}{n} x^n \cdot \frac{1}{x^n} = \binom{2n}{n}$$

is independent of x, and is therefore a constant.

IP1:

If the 3rd term of the expansion $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000, then the value of x is

Solution:

$$Given\ \ T_3 = T_{2+1} = 1000$$

$$\Rightarrow {}^5C_2\left(\frac{1}{x}\right)^{5-2}\left(x^{\log_{10} x}\right)^2 = 1000$$

$$\Rightarrow 10x^{-3}\left(x^{\log_{10} x}\right)^2 = 1000$$

$$\Rightarrow x^{2\log_{10} x - 3} = 10^2$$

$$\Rightarrow 2\log_{10} x - 3 = 2\log_x 10$$

Put $\log_{10} x = y$, we get

$$2y - 3 = \frac{2}{y} \Rightarrow 2y^2 - 3y - 2 = 0 \Rightarrow (y-2)(2y+1) = 0$$

$$\Rightarrow y = 2 \text{ or } y = \frac{-1}{2}$$

$$\Rightarrow \log_{10} x = 2 \text{ or } \log_{10} x = \frac{-1}{2} \Rightarrow x = 10^2 \text{ or } x = 10^{-\frac{1}{2}}$$

$$\Rightarrow x = 100 \text{ or } x = \frac{1}{\sqrt{10}}$$

IP2:

Find the coefficients of the middle terms of the expansion

$\left(3x-\frac{x^3}{6}\right)^7$.

Solution:

We have $n=7$ (odd). The middle terms are $\frac{n+1}{2}=4, \frac{n+3}{2}=5$

The middle terms of the given expansion $\left(3x-\frac{x^3}{6}\right)^7$ are 4th and 5th terms

$$T_4 = T_{3+1} = {}^7C_3\,(3x)^{7-3}\left(-\frac{x^3}{6}\right)^3$$

$$= {}^7C_3\,(-1)^3.3^4.x^4..x^9.6^{-3}$$

$$= -35\left(81x^4\right).\left(\frac{x^9}{216}\right) = -\frac{105}{8}x^{13}$$

$$T_5 = T_{4+1} = {}^7C_4\,(3x)^{7-4}\left(-\frac{x^3}{6}\right)^4$$

$$= {}^7C_4.3^3.x^3..x^{12}.6^{-4}$$

$$= \frac{35(27)}{1296}x^{15} = \frac{35}{48}x^{15}$$

IP3:

If the coefficient of x in the expansion $\left(x^2 + \frac{k}{x}\right)^5$ is 270 then $k =$

Solution:

The given expansion is $\left(x^2 + \frac{k}{x}\right)^5$

Now, general term is

$$T_{r+1} = {}^5C_r \left(x^2\right)^{5-r} \left(\frac{k}{x}\right)^r = {}^5C_r\, x^{10-3r}\, k^r$$

To find the coefficient of x, we must have $10 - 3r = 1$

$$\Rightarrow r = 3$$

$$\therefore\ T_{3+1} = {}^5C_3\ k^3 x$$

By the hypothesis, the coefficient of x is 270

$$\frac{5!}{2!3!} k^3 = 270$$

$$\Rightarrow k^3(10) = 270 \Rightarrow k = 3$$

IP4:

The sum of the coefficients of x^m, x^{m-3}, x^{m-6} in the expansion of $\left(x-\frac{3}{x^2}\right)^m, x \neq 0$, ($m$ being a natural number) is 559. Find the coefficient of x^3 in the expansion.

Solution:

The coefficients of x^m, x^{m-3}, x^{m-6} in the expansion of $\left(x-\frac{3}{x^2}\right)^m, x \neq 0$, ($m$ being a natural number), are ${}^mC_0$, ${}^mC_1(-3)$ and ${}^mC_2(9)$.

By hypothesis, we have

$${}^mC_0 - 3\,{}^mC_1 + 9\,{}^mC_2 = 559$$

$$\Rightarrow 1 - 3m + \frac{9m(m-1)}{2} = 559$$

$$\Rightarrow 2 - 6m + 9m^2 - 9m = 1118$$

$$\Rightarrow 3m^2 - 5m - 372 = 0$$

$$\Rightarrow (3m+31)(m-12) = 0$$

$$\Rightarrow m = 12 \ \ (m \text{ being a natural number})$$

Now,

$$T_{r+1} = {}^{12}C_r\, x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r\,(-3)^r\, x^{12-3r}$$

To compute the term containing x^3, put $12 - 3r = 3 \Rightarrow r = 3$

Thus, the required term is ${}^{12}C_3\,(-3)^3\, x^{12-9} = -5940x^3$.

P1:

If the 2nd, 3rd and 4th terms in the binomial expansion $(x+a)^n$ are 240, 720 and 1080 respectively, then find x, a and n.

Solution:

Given

$$T_2 = T_{1+1} = {}^nC_1\, x^{n-1}.a = 240 \ \ ....(A)$$
$$T_3 = T_{2+1} = {}^nC_2\, x^{n-2}.a^2 = 720..........(B)$$
$$T_4 = T_{3+1} = {}^nC_3\, x^{n-3}.a^3 = 1080.........(C)$$

$$\frac{(B)}{(A)} \Rightarrow \frac{{}^nC_2\, x^{n-2}.a^2}{{}^nC_1\, x^{n-1}.a} = \frac{720}{240}$$

$$\Rightarrow \frac{\frac{n(n-1)}{2!}}{n}.\frac{a}{x} = 3 \Rightarrow \frac{a}{x} = \frac{6}{(n-1)}..........(1)$$

$$\frac{(C)}{(B)} \Rightarrow \frac{{}^nC_3\, x^{n-3}.a^3}{{}^nC_2\, x^{n-2}.a^2} = \frac{1080}{720}$$

$$\Rightarrow \frac{(n-2)}{3}.\frac{a}{x} = \frac{3}{2} \Rightarrow \frac{a}{x} = \frac{9}{2(n-2)}..........(2)$$

By solving (1) and (2), we get

$$\frac{6}{(n-1)} = \frac{9}{2(n-2)} \Longrightarrow n = 5$$

From (1), we have $a = \frac{3}{2}x$

From (A), we have

$${}^5C_1\, x^4 a = 240 \Rightarrow 5x^4\left(\frac{3x}{2}\right) = 240 \Rightarrow x^5 = 32 \Rightarrow x = 2$$

and $a = \frac{3}{2}(2) \Longrightarrow a = 3$

Therefore, the values of $x,\ a,\ n$ are $2,\ 3,\ 5$ respectively.

P2:

The middle term in the expansion of $\left(\frac{3}{x^3}+5x^4\right)^{20}$ is

Solution:

Here $n = 20$. There exists only one middle term, since n is even.

Middle term $= \left(\frac{n}{2} + 1\right)^{th} term$

$$= \left(\frac{20}{2} + 1\right)^{th} term = 11^{\text{th}} \text{ term}$$

$$T_{11} = T_{10+1} = {}^{20}C_{10}\left(\frac{3}{x^3}\right)^{20-10}\left(5x^4\right)^{10}$$

$$= {}^{20}C_{10} 15^{10} x^{10}$$

P3:

The independent term of x in $\left(2x^{\frac{1}{2}} - 3x^{-\frac{1}{3}}\right)^{20}$ is

Solution:

The given expansion is $\left(2x^{\frac{1}{2}} - 3x^{-\frac{1}{3}}\right)^{20}$

$$\therefore\ T_{r+1} = {}^{20}C_r\ (2x^{1/2})^{20-r}\,(-3x^{-1/3})^r$$

$$= {}^{20}C_r\, 2^{20-r} 3^r\, x^{\frac{20-r}{2}} (-1)^r\, x^{-\frac{r}{3}}$$

$$= {}^{20}C_r\, (-1)^r\, 2^{20-r}\, 3^r\, x^{10-\frac{r}{2}-\frac{r}{3}}$$

To, find the independent term, we must have

$$10 - \frac{r}{2} - \frac{r}{3} = 0 \Rightarrow 10 = \frac{5r}{6} \Rightarrow r = \frac{60}{5} \Rightarrow r = 12$$

$$\therefore\ T_{12+1} = {}^{20}C_{12}\,(-1)^{12}\, 2^{20-12}\, 3^{12} = {}^{20}C_{12}\, 2^8\ 3^{12}$$

P4:

If the coefficients of 5^{th}, 6^{th}, 7^{th} terms of $(1+x)^n$ are in A. P then $n =$

Solution:

The coefficients of 5th, 6th and 7th terms of $(1+x)$ are

$$^nC_4, {}^nC_5 \ and \ {}^nC_6$$

Given, $^nC_4, {}^nC_5 \ and \ {}^nC_6$ are in A.P.

$$\Rightarrow 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$$

$$\Rightarrow 2.\frac{n!}{(n-5)!5!} = \frac{n!}{(n-4)!4!} + \frac{n!}{(n-6)!6!}$$

$$\Rightarrow \quad \frac{2}{(n-5)(n-6)!5.4!} = \frac{1}{(n-4)(n-5)(n-6)!4!} + \frac{1}{(n-6)!6.5.4!}$$

$$\Rightarrow \quad \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{30}$$

$$\Rightarrow \quad \frac{1}{n-5}\left[\frac{2}{5} - \frac{1}{n-4}\right] = \frac{1}{30}$$

$$\Rightarrow \quad \frac{2n-8-5}{5(n-4)} = \frac{n-5}{30}$$

$$\Rightarrow \quad 12n - 48 - 30 = n^2 - 9n + 20$$

$$\Rightarrow \quad n^2 - 21n + 98 = 0$$

$$\Rightarrow \quad (n-14)(n-7) = 0 \Rightarrow \quad n = 7 \text{ or } 14$$

Exercises:

1. Write down and simplify

   a. $5th$ term in $(3x - 4y)^7$

   b. $6th$ term in $\left(\frac{2x}{3} + \frac{3y}{2}\right)^9$

   c. $7th$ term in $(3x - 4y)^{10}$

   d. $7th$ term in $\left(\frac{4}{x^3} + \frac{x^2}{2}\right)^{12}$

2. Find the twelfth term of $\left(\frac{x^{1/2}}{4} - \frac{2y}{x^{3/2}}\right)^{18}$ and simplify.

3. Find a if the 17^{th} and 18^{th} terms of the expansion $(2 + a)^{50}$ are equal.

4. Find the middle term(s) in the expansion of

a. $\left(\frac{3x}{7} - 2y\right)^{10}$

b. $(3a - 2b)^6$

c. $\left(4a + \frac{3}{2}b\right)^{11}$

d. $(2x + 3y)^7$

5. Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} 2^n x^n$, where n is a positive integer.

6. If the k th term is the middle term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$, then find T_k and T_{k+3}.

7. Find the term independent of x in the expansion of

a. $\left(\frac{x^{\frac{1}{2}}}{3} - \frac{4}{x^2}\right)$

b. $\left(\frac{3}{\sqrt[3]{x}} + 5\sqrt{x}\right)^{25}$

c. $\left(4x^3 + \frac{7}{x^2}\right)^{14}$

d. $\left(\frac{2x^2}{5} + \frac{15}{4x}\right)^9$

e. $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$

8. Find the coefficient of x^6y^3 in the expansion of $(x+2y)^9$.

9. The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio $1:7:42$. Find n.

10. Show that the coefficient of the middle term in the expansion of $(1 + 2x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

11. Find the coefficient of a^4 in the product $(1+2a)^4(2-a)^5$ using binomial theorem.

12. If the coefficients of $(r-5)^{\text{th}}$ and $(2r-1)^{\text{th}}$ terms in the expansion of $(1+x)^{34}$ are equal, then find r.

7.3

Greatest Coefficient and Greatest Term

Learning objectives:

- To find the greatest coefficients and greatest terms in the binomial expansion.
  And
- To practice the related problems.

Greatest Coefficient

In any binomial expansion middle term has the greatest coefficient. If there are two middle terms, then their coefficients are equal and greatest.

We consider the binomial expansion of $(x+a)^n$; let ${}^nC_r$ be the greatest coefficient.

Now, $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{n!}{r!(n-r)!} \cdot \frac{(r+1)!(n-r-1)!}{n!} = \frac{r+1}{n-r}$ -------(1)

Since ${}^nC_r$ is the greatest coefficient

$\frac{r+1}{n-r} \geq 1 \Rightarrow r+1 \geq n-r \Rightarrow r \geq \frac{n-1}{2}$ -------(2)

If we substitute $r-1$ for r in (1), we get

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-r+1}$$

Since ${}^nC_r$ is the greatest coefficient

$\frac{r}{n-r+1} \leq 1 \Rightarrow r \leq n-r+1 \Rightarrow r \leq \frac{n+1}{2}$ ------(3)

From (2) and (3), we have

$$\frac{n-1}{2} \leq r \leq \frac{n+1}{2} \qquad \text{--------}(4)$$

When $n = 2m$ (an even number)

$$m - \frac{1}{2} \leq r \leq m + \frac{1}{2}$$

This implies $r = m$, and the greatest coefficient is the coefficient of the middle term.

When $n = 2m+1$ (an odd number)

$$m \leq r \leq m+1$$

This implies $r = m, m+1$

In the next module, we show that the coefficients of the terms in the binomial expansion of $(x+a)^n$ equidistant from the beginning and the end are equal. So, the greatest coefficient is the coefficient of the two middle terms.

Example

In the expansion of $(x+y)^4$, there are 5 terms. The middle term is 3. The greatest coefficient is the coefficient of the 3rd term

$$T_3 = {}^4C_2 x^2 y^2 = 6x^2y^2$$

Example:

In the expansion of $(x+y)^5$, there are 6 terms. There are two middle terms given by $\frac{5+1}{2} = 3$ and $\frac{5+3}{2} = 4$. The two middle terms are 3rd and 4th terms.

Their coefficients are greatest and equal.

$T_3 = {}^5C_2 x^2 y^3 = 10x^2y^3$, $T_4 = {}^5C_3 x^3 y^2 = 10x^3y^2$

Greatest Term

In a binomial expansion, greatest term means numerically greatest term. Since we are only concerned with the numerically greatest term, the investigation will be the same for $(x-a)^n$ as for $(x+a)^n$. Therefore, in any numerical example it is unnecessary to consider the sign of the second term of the binomial.

We write $(x+a)^n = x^n\left(1+\frac{a}{x}\right)^n$

Since x^n multiplies every term in $\left(1+\frac{a}{x}\right)^n$, it will be sufficient to find the greatest term in this latter expansion.

Let the r^{th} and $(r+1)^{\text{th}}$ be any two consecutive terms. The $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\frac{n-r+1}{r} \cdot \frac{a}{x}$; that is, by $\left(\frac{n+1}{r}-1\right)\frac{a}{x}$.

The factor $\left(\frac{n+1}{r}-1\right)$ decreases as r increases; hence the $(r+1)^{\text{th}}$ term is not always greater than the r^{th} term, but only until $\left(\frac{n+1}{r}-1\right)\frac{a}{x}$ becomes equal to 1, or less than 1.

Now

$$\left(\frac{n+1}{r}-1\right)\frac{a}{x} > 1 \Rightarrow \left(\frac{n+1}{r}-1\right) > \frac{x}{a}$$

$$\Rightarrow \frac{n+1}{r} > \frac{x}{a}+1 \Rightarrow \frac{n+1}{\frac{x}{a}+1} > r \qquad \text{--------}(5)$$

If $\frac{n+1}{\frac{x}{a}+1}$ is an integer, we denote it by p; then if $r = p$, the multiplying factor becomes 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th}; and these are greater than any other term.

If $\frac{n+1}{\frac{x}{a}+1}$ is not an integer, we denote its integral part by q; then the greatest value of r consistent with (5) is q; hence the $(q+1)^{\text{th}}$ term is the greatest.

Example:

If $x = \frac{1}{3}$, find the greatest term in the expansion of $(1+4x)^8$.

Solution

Here $n = 8$, $\frac{a}{x} = 4x \Rightarrow \frac{x}{a} = \frac{1}{4x} = \frac{3}{4}$ $\left(\because x = \frac{1}{3}\right)$

Now, $\frac{n+1}{\frac{x}{a}+1} = \frac{8+1}{\frac{3}{4}+1} = \frac{36}{7}$, not an integer.

Therefore, its integral part $q = 5$ and

$(q+1) = 5+1 = 6^{th}$ term is greatest term and its value is

$$T_6 = T_{5+1} = {}^8C_5\left(\frac{4}{3}\right)^5$$

Example:

Find the greatest term in the expansion of $(3-2x)^9$ when $x = 1$.

Solution:

Given $(3-2x)^9 = 3^9\left(1-\frac{2x}{3}\right)^9$

Here $n = 9$, $\frac{a}{x} = \frac{2x}{3}$ (neglecting the sign)

$\Rightarrow \frac{x}{a} = \frac{3}{2x} = \frac{3}{2} (\because x = 1)$

Now, $\frac{n+1}{\frac{x}{a}+1} = \frac{9+1}{\frac{3}{2}+1} = \frac{20}{5} = 4$, is an integer.

$\therefore$ 4th and 5th terms are greatest terms, which are numerically equal and its value is

$$3^9 \; {}^9C_3\left(\frac{2}{3}\right)^3 = 3^6 \times 84 \times 8 = 4{,}89{,}888$$

IP1:

The greatest Binomial Coefficient in the expansion $(x+y)^{105}$ is

Solution:

The given expansion is $(x+y)^{105}$

Here index $n = 105$, which is odd

$\therefore$ The greatest coefficients are ${}^{n}C_{\frac{n-1}{2}}$ and ${}^{n}C_{\frac{n+1}{2}}$

i.e. ${}^{105}C_{\frac{105-1}{2}}$ and ${}^{105}C_{\frac{105+1}{2}}$

$\Rightarrow$ ${}^{105}C_{\frac{104}{2}}$ and ${}^{105}C_{\frac{106}{2}}$

$\Rightarrow$ ${}^{105}C_{52}$ and ${}^{105}C_{53}$

Notice that ${}^{105}C_{52} = {}^{105}C_{53}$ (since ${}^{n}C_{r} = {}^{n}C_{n-r}$)

IP2:

Find the numerically greatest term in the expansion of $(1-5x)^{12}$ when $x=\frac{2}{3}$.

Solution:

The given expansion is $(1-5x)^{12}$

Here $n=12$, $\frac{a}{x}=5x$ (neglecting the sign)

$$\Longrightarrow \frac{x}{a}=\frac{1}{5x}=\frac{3}{10} \quad \left(\because x=\frac{2}{3}\right)$$

Now, $\frac{n+1}{\frac{x}{a}+1}=\frac{12+1}{\frac{3}{10}+1}=10$, is an integer.

$\therefore$ 10^{th} and 11^{th} terms are the greatest terms which are numerically equal and its value is

$$^{12}C_{10}\times\left(\frac{10}{3}\right)^{10}$$

IP3:

Find the numerically greatest term in the expansion $(3x - 4y)^{14}$ when $x = 8$ and $y = 3$.

Solution:

Given $(3x - 4y)^{14} = (3x)^{14}\left(1 - \frac{4y}{3x}\right)^{14}$

Here $n = 14$, $\frac{a}{x} = \frac{4y}{3x}$ (neglecting the sign)

$\Longrightarrow \frac{x}{a} = \frac{3x}{4y} = 2(\because x = 8 \; and \; y = 3)$

Now, $\frac{n+1}{\frac{x}{a}+1} = \frac{14+1}{2+1} = \frac{15}{3} = 5$, is an integer.

$\therefore$ 5^{th} and 6^{th} terms are greatest terms which are numerically equal and its value is

$$(24)^{14} \times {}^{14}C_4\left(\frac{1}{2}\right)^4$$

IP4:

Find the numerically greatest term in the expansion of $(3x+2y)^{11}$ when $x=\frac{2}{3}$ and $y=\frac{3}{4}$.

Solution:

Here $n=11$, $\frac{a}{x}=\frac{2y}{3x} \Longrightarrow \frac{x}{a}=\frac{3x}{2y}=\frac{4}{3}\left(\because x=\frac{2}{3}, y=\frac{3}{4}\right)$

Now, $\frac{n+1}{\frac{x}{a}+1}=\frac{11+1}{\frac{4}{3}+1}=\frac{36}{7}$, is not an integer.

Therefore, its integral part is $q=5$

$\therefore\ q+1=6^{\text{th}}$ term is the greatest term and its value is

$$(3x)^{11}\cdot {}^{11}C_5\left(\frac{2y}{3x}\right)^5={}^{11}C_5(2y)^5(3x)^6={}^{11}C_6(6)^5(2)^6$$

P1:

If ${}^{22}C_r$ is the greatest Binomial Coefficient in the expansion $(1+x)^{22}$ then ${}^{13}C_r =$

Solution:

The given expansion is $(1+x)^{22}$

Here index $n = 22$, which is even

$\therefore$ The greatest coefficient is ${}^{n}C_{\frac{n}{2}}$ i.e. ${}^{22}C_{\frac{22}{2}} = {}^{22}C_{11}$

By the hypothesis, ${}^{22}C_{r} = {}^{22}C_{11} \Rightarrow r = 11$

Now, ${}^{13}C_{r} = {}^{13}C_{11} = \dfrac{13!}{11!\,2!} = \dfrac{13 \times 12}{2} = 78$

P2:

Find the numerically greatest term in the expansion

$\left(3+\frac{2x}{5}\right)^{12}$ when $x=\frac{3}{4}$.

Solution:

Given $\left(3+\frac{2x}{5}\right)^{12}=3^{12}\left(1+\frac{2x}{15}\right)^{12}$

Here $n=12$, $\frac{a}{x}=\frac{2x}{15}\Longrightarrow\frac{x}{a}=\frac{15}{2x}=10\left(\because x=\frac{3}{4}\right)$

Now, $\frac{n+1}{\frac{x}{a}+1}=\frac{12+1}{10+1}=\frac{13}{11}$, is not an integer.

Therefore, its integral part is $q=1$

$\therefore\ q+1=2^{nd}$ term is the greatest term and its value is

$$T_2=3^{12}\times{}^{12}C_1\times\left(\frac{1}{10}\right)=\frac{6}{5}\times3^{12}$$

P3:

Find the numerically greatest term in the expansion of $(2x-3y)^{12}$ when $x=1$ and $=\frac{5}{3}$.

Solution:

Given $(2x - 3y)^{12} = (2x)^{12}\left(1 - \frac{3y}{2x}\right)^{12}$

Here $n = 12$, $\frac{a}{x} = \frac{3y}{2x}$ (neglecting the sign)

$\Longrightarrow \frac{x}{a} = \frac{2x}{3y} = \frac{2}{5}\left(\because x = 1\ and\ y = \frac{5}{3}\right)$

Now, $\frac{n+1}{\frac{x}{a}+1} = \frac{12+1}{\frac{2}{5}+1} = \frac{65}{7}$ is not an integer

Therefore, its integral part is $q = 9$

$\therefore$ $q + 1 = 10^{th}$ term is the greatest term and its value is

$$(2x)^{12} \times {}^{12}C_9 \times \left(\frac{5}{2}\right)^9 = {}^{12}C_9 \times 2^3 \times 5^9$$

P4:

Find the numerically greatest term in the expansion of $(3x-5y)^n$ when $x=\frac{3}{4}$, $y=\frac{2}{7}$ and $n=17$.

Solution:

First write $(3x-5y)^n=(3x-5y)^{17}=(3x)^{17}\left(1-\frac{5y}{3x}\right)^{17}$

Here $n=17$, $\quad \frac{a}{x}=\frac{5y}{3x}$ (neglecting the sign)

$\Longrightarrow \frac{x}{a}=\frac{3x}{5y}=\frac{63}{40}$

Now, $\frac{(n+1)}{\frac{x}{a}+1}=\frac{17+1}{\frac{63}{40}+1}=\frac{720}{103}$, which not an integer

Therefore, its integral part is $q=6$ and its greatest term is 7th term.

$$(3x)^{17}\cdot T_7=(3x)^{17}\cdot {}^{17}C_6\left(\frac{5y}{3x}\right)^6$$

$$={}^{17}C_6(5y)^6(3x)^{11}$$

$$={}^{17}C_6\left(\frac{10}{7}\right)^6\left(\frac{9}{4}\right)^{11}$$

Exercises:

1. Find the greatest binomial coefficient(s) in the expansion of

a. $(1+x)^{19}$

b. $(1+x)^{24}$

c. $\left(x+\frac{1}{x}\right)^{2n}$

2. Find the numerically greatest term(s) in the expansion of

a. $(2+3x)^{10}$ when $x = \frac{11}{8}$

b. $(4+3x)^{15}$ when $x = \frac{7}{2}$

c. $(7-5x)^{11}$ where $x = \frac{2}{3}$

d. $(1-3x)^{10}$ when $x = \frac{1}{2}$

e. $(3x+5y)^{12}$ when $x = \frac{1}{2}, y = \frac{4}{3}$

f. $(3y+7x)^{10}$ when $y = \frac{1}{2}, x = \frac{1}{3}$

g. $(4a-6b)^{13}$ when $a = 3, b = 5$

h. $(3+7x)^{n}$ when $x = \frac{4}{5}, n = 15$

i. $(3x-4y)^{14}$ when $x = 8, y = 3$

8.4

Binomial Coefficients

Learning objectives:

- To derive some properties of binomial coefficients. And
- To practice the related problems.

The values of ${}^nC_r$ are often referred to as *binomial coefficients*. This is so because of their prominence in the binomial theorem.
Some properties of the binomial coefficients are derived below.
The binomial coefficients are defined by

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

We note that $${}^nC_{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{r!(n-r)!}$$

Therefore, $${}^n\mathbf{C}_{n-r} = {}^n\mathbf{C}_r$$

If $^n\mathbf{C}_x = {}^n\mathbf{C}_y$***, then either*** $\boldsymbol{x = y}$ ***or*** $\boldsymbol{x + y = n}$.
The first relation readily follows from the formulas of the left and right hand sides.
For the second relation, we note that ${}^nC_x = {}^nC_y = {}^nC_{n-y}$
Therefore, $x = n - y \implies x + y = n$

The following relations can be easily obtained from the formula for ${}^nC_r$.

$$\mathbf{r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}}$$

$$\mathbf{{}^nC_r = \frac{r+1}{n+1} \cdot {}^{n+1}C_{r+1}}$$

A useful combinatorial identity (called Pascal's identity) is

$$\mathbf{{}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad 1 \le r \le n} \text{------}(1)$$

Equation (1) may be proved analytically or by the following combinatorial argument. Consider a group of n objects and fix attention on some particular one of these objects – call it object 1. Now, there are ${}^{n-1}C_{r-1}$ groups of size r that contain object 1 (since each such group is formed by selecting $r - 1$ from the remaining $n - 1$ objects). Also there are ${}^{n-1}C_r$ groups of size r that do not contain object 1. As there is a total of ${}^nC_r$ groups of size r, equation (1) follows.

The coefficients of the terms equidistant from the beginning and the end are equal.
This is seen as follows.
The coefficient of the $(r + 1)^{\text{th}}$ term from the beginning is ${}^nC_r$. There are altogether $(n + 1)$ terms and the $(r + 1)^{\text{th}}$ term from the end has $(n + 1) - (r + 1)$ or $(n - r)$ terms before it. Hence the $(r + 1)^{\text{th}}$ term from the end is $(n - r + 1)^{\text{th}}$ term from the beginning and its coefficient is ${}^nC_{n-r}$.
Since ${}^nC_r = {}^nC_{n-r}$, the assertion is proved.

We put $x = 1$ in the expansion of

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n \text{ --------}(2)$$

we get

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$$

Therefore, the sum of the binomial coefficients is 2^n.
It follows that

$${}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n - 1$$

If we put $x = -1$ in the expansion (2), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \cdots = 0$$

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$$
$$= \frac{1}{2}(\text{sum of all the coefficients})$$
$$= \frac{2^n}{2} = 2^{n-1}$$

The sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms, and each is equal to 2^{n-1}.

Example 1:

$${}^{10}C_0 + {}^{10}C_2 + {}^{10}C_4 + \cdots + {}^{10}C_{10} = 2^{10-1} = 2^9$$
$${}^{11}C_0 + {}^{11}C_2 + {}^{11}C_4 + \cdots + {}^{11}C_{10} = 2^{11-1} = 2^{10}$$
$${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + \cdots + {}^{10}C_9 = 2^{10-1} = 2^9$$
$${}^{11}C_1 + {}^{11}C_3 + {}^{11}C_5 + \cdots + {}^{11}C_{11} = 2^{11-1} = 2^{10}$$

Example 2:

If in the expansion of $(1 + x)^{43}$, the coefficient of $(2r + 1)^{\text{th}}$ term is equal to the coefficient of $(r + 2)^{\text{th}}$ term, then find r.
Solution:

$${}^{43}C_{2r} = {}^{43}C_{r+1}$$

Therefore, $2r + r + 1 = 43 \implies r = 14$

Example 3:

A man has 6 friends; in how many ways may he invite one or more of them to dinner?
Solution:
The guests may be invited singly, in twos, threes…; therefore the number of selection

$${}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6 = 2^6 - 1 = 63$$

IP1:

If the coefficients of $(\boldsymbol{r-5})^{th}$ and $(\boldsymbol{2r-1})^{th}$ terms in the expansion of $(\boldsymbol{1+x})^{34}$ are equal then the value of $\boldsymbol{r}$ is

Solution:

The coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are ${}^{34}C_{r-6}$ *and* ${}^{34}C_{2r-2}$

By the hypothesis, we have ${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

Therefore, either $r-6 = 2r-2$ or $r-6+(2r-2) = 34$

$\Rightarrow r = -4 \ or \ r = 14$

$\therefore r = 14$ ($\because r$ is a positive integer)

IP2:

Prove that

i) $$ {}^{n}C_0 + 3\cdot {}^{n}C_1 + 3^2\cdot {}^{n}C_2 + \cdots + 3^n\cdot {}^{n}C_n = 4^n $$

ii) $$ \frac{{}^{n}C_1}{{}^{n}C_0} + 2\cdot\frac{{}^{n}C_2}{{}^{n}C_1} + 3\cdot\frac{{}^{n}C_3}{{}^{n}C_2} + \cdots + n\cdot\frac{{}^{n}C_n}{{}^{n}C_{n-1}} = \frac{n(n+1)}{2} $$

Solution:

i) We know that

$$ (1+x)^n = {}^{n}C_0 + {}^{n}C_1\cdot x + {}^{n}C_2\cdot x^2 + \cdots + {}^{n}C_n\cdot x^n $$

Put $x = 3$, we get

$$ {}^{n}C_0 + 3\cdot {}^{n}C_1 + 3^2\cdot {}^{n}C_2 + \cdots + 3^n\cdot {}^{n}C_n = 4^n $$

ii)

$$ \frac{{}^{n}C_1}{{}^{n}C_0} + 2\cdot\frac{{}^{n}C_2}{{}^{n}C_1} + 3\cdot\frac{{}^{n}C_3}{{}^{n}C_2} + \cdots + n\cdot\frac{{}^{n}C_n}{{}^{n}C_{n-1}} = \sum_{r=1}^{n} r\cdot\frac{{}^{n}C_r}{{}^{n}C_{r-1}} $$

$$ = \sum_{r=1}^{n} r\cdot\frac{n!}{(n-r)!r!}\times\frac{(n-r+1)!(r-1)!}{n!} $$

$$ = \sum_{r=1}^{n} (n-r+1) $$

$$ = n + (n-1) + (n-2) + \cdots + 2 + 1 $$

$$ = \frac{n(n+1)}{2} $$

IP3:

$$\left({}^{2n}C_0\right)^2-\left({}^{2n}C_1\right)^2+\left({}^{2n}C_2\right)^2- \;\dots\dots\dots+\left({}^{2n}C_{2n}\right)^2=$$

Solution:

We have

$$(1+x)^{2n}={}^{2n}C_0+{}^{2n}C_1\,x+\dots\dots\dots+{}^{2n}C_{2n}\,x^{2n} \text{ ---------- } (1)$$

$$(x-1)^{2n}={}^{2n}C_1x^{2n}-{}^{2n}C_1x^{2n-1}+\dots\dots\dots+{}^{2n}C_{2n}\text{------ } (2)$$

Multiplying (1) & (2) we get

$$\left(x^2-1\right)^{2n}=\left({}^{2n}C_0x^{2n}-{}^{2n}C_1\,x^{2n-1}+{}^{2n}C_2\,x^{2n-2}-\;\dots\dots+{}^{2n}C_{2n}\right)\cdot$$
$$\left({}^{2n}C_0+{}^{2n}C_1\,x+{}^{2n}C_2\,x^2+\;\dots\dots+{}^{2n}C_{2n}\,x^{2n}\right)$$

By comparing the x^{2n} coefficients on both sides, we get

$$\left({}^{2n}C_0\right)^2-\left({}^{2n}C_1\right)^2+\left({}^{2n}C_2\right)^2-\cdots+\left({}^{2n}C_{2n}\right)^2={}^{2n}C_n(-1)^n$$

IP4:

If C_r denotes ^nC_r, then prove that

$$\mathbf{3 \cdot C_0^2 + 7 \cdot C_1^2 + 11 \cdot C_2^2 + \cdots + (4n+3)C_n^2 = (2n+3) \cdot {}^{2n}C_n}.$$

Solution:

Let $S = 3 \cdot C_0^2 + 7 \cdot C_1^2 + 11 \cdot C_2^2 + \cdots + (4n+3)C_n^2$ ------ (1)

On writing the terms of the $R.H.S.$ of (1), in the reverse order, we get

$$S = (4n+3)C_n^2 + (4n-1)C_{n-1}^2 + (4n-5)C_{n-2}^2 + \cdots + 3C_0^2$$

$$= (4n+3)C_0^2 + (4n-1)C_1^2 + (4n-5)C_2^2 + \cdots + 3C_n^2 \text{ --(2)}$$

$$(\because\ C_{n-r} = C_r, for\ 0 \le r \le n)$$

On adding (1) & (2), we get

$$2S = (4n+6)C_0^2 + (4n+6)C_1^2 + (4n+6)C_2^2 + \cdots + (4n+6)C_n^2$$

$$= (4n+6)\left(C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2\right)$$

$$= (4n+6) \cdot {}^{2n}C_n \quad \left(\because C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = {}^{2n}C_n\right)$$

$$\therefore S = (2n+3) \cdot {}^{2n}C_n$$

P1:

If the coefficients of $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal then $r =$

Solution:

The $(2r+4)^{th}$ term of the given expansion $(1+x)^{18}$ is

$$T_{2r+4} = T_{(2r+3)+1} = {}^{18}C_{2r+3} \cdot x^{15-2r}$$

Thus, the coefficient of $(2r+4)^{th}$ term is ${}^{18}C_{2r+3}$

Similarly, the coefficient of $(r-2)^{th}$ is ${}^{18}C_{r-3}$

By the hypothesis, ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\implies$ $2r+3 = r-3$ or $(2r+3)+(r-3) = 18$

$\implies$ $r = -6$ or $r = 6$

$\therefore r = 6$ (Since r is a positive integer)

P2:

If n is a positive integer, prove that

i. $\sum_{r=1}^{n} r \cdot {}^{n}C_{r} = n \cdot 2^{n-1}$

ii. $\sum_{r=2}^{n} r(r-1) \cdot {}^{n}C_{r} = n(n-1) \cdot 2^{n-2}$

iii. $\sum_{r=1}^{n} r^{2} \cdot {}^{n}C_{r} = n(n+1) \cdot 2^{n-2}$

Solution:

i. We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_n \cdot x^n$$

On differentiating both sides w.r.t x, we get

$$n(1+x)^{n-1} = {}^nC_1 + {}^nC_2 \cdot 2x + {}^nC_3 \cdot 3x^2 + \cdots + {}^nC_n \cdot nx^{n-1}$$

Now put $x = 1$, we get

$$n \cdot 2^{n-1} = {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \cdots + n \cdot {}^nC_n$$

Thus, $\sum\limits_{r=1}^{n} r \cdot {}^nC_r = n \cdot 2^{n-1}$

ii. We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_n \cdot x^n$$

On differentiating both sides w.r.t x, we get

$$n(1+x)^{n-1} = {}^nC_1 + {}^nC_2 \cdot 2x + {}^nC_3 \cdot 3x^2 + \cdots + {}^nC_n \cdot nx^{n-1}$$

Again differentiating both sides w.r.t x, we get

$$n(n-1)(1+x)^{n-2} = 2\,{}^nC_2 + {}^nC_3 \cdot 6x + \cdots + {}^nC_n \cdot n(n-1)x^{n-2}$$

Now put $x = 1$, we get

$$n(n-1)2^{n-2} = 2\,{}^nC_2 + 6\,{}^nC_3 + \cdots + n(n-1)\,{}^nC_n$$

Thus $\sum\limits_{r=2}^{n} r(r-1) \cdot {}^nC_r = n(n-1) \cdot 2^{n-2}$

iii. $\sum\limits_{r=1}^{n} r^2 \cdot {}^nC_r = \sum\limits_{r=1}^{n} \left[r(r-1)+r\right] {}^nC_r$

$$= \sum_{r=1}^{n} r(r-1) \cdot {}^nC_r + \sum_{r=1}^{n} r \cdot {}^nC_r$$

$$= n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1}$$

$$= (n-1+2)n \cdot 2^{n-2}$$

P3:

$$C_0 + \frac{3}{2}C_1 + \frac{9}{3}C_2 + \frac{27}{4}C_3 + \ldots\ldots + \frac{3^n}{n+1}C_n =$$

Solution:

We have $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$

By integrating, we get

$$\frac{(1+x)^{n+1}}{n+1} = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \cdots + \frac{C_n x^{n+1}}{n+1}$$

Applying the limits 0 to 3, we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \cdots + \frac{C_n x^{n+1}}{n+1}\right]_0^3$$

$$\left[\frac{4^{n+1}}{n+1} - \frac{1}{n+1}\right] = 3\left[C_0 + \frac{3}{2}C_1 + \frac{9}{3}C_2 + \cdots + \frac{3^n}{n+1}C_n\right]$$

$$\frac{4^{n+1}-1}{3(n+1)} = C_0 + \frac{3}{2}C_1 + \frac{9}{3}C_2 + \cdots + \frac{3^n}{n+1}C_n$$

P4:

If C_r denotes ${}^nC_r$, then prove that

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n = {}^{2n}C_{n+r}$$

and deduce the following:

1. $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n$
2. $C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots + C_{n-1}C_n = {}^{2n}C_{n+1}$

Solution:

We have

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n \qquad \text{------ (1)}$$

and $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_n}{x^n}$ ------ (2)

Multiplying (1) & (2) we get

$$\frac{(1+x)^{2n}}{x^n} = \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_n}{x^n}\right)$$

$$(C_0 + C_1x + C_2x^2 + \cdots + C_nx^n) \qquad \text{------(3)}$$

The coefficient of x^r on RHS of (3) is

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n$$

The coefficient of x^r on LHS

$= \text{The coefficient of } x^r \text{ in } \frac{(1+x)^{2n}}{x^n}$

$= \text{The coefficient of } x^{n+r} \text{ in } (1+x)^{2n}$

$= {}^{2n}C_{n+r}$

Comparing the coefficient of x^r on both sides of (3), we get

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n = {}^{2n}C_{n+r} \qquad \text{-------(4)}$$

Deduction 1: Put $r = 0$ in (4), we get

$$C_0{}^2 + C_1{}^2 + C_2{}^2 + \cdots + C_n{}^2 = {}^{2n}C_n$$

Deduction 2: Put $r = 1$ in (4), we get

$$C_0C_1 + C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n = 2n_{C_{n+1}}$$

1. Prove that $C_0 + 2 \cdot C_1 + 2^2 \cdot C_2 + \cdots + 2^n \cdot C_n = 3^n$.

2. Show that

i. $2 \cdot C_0 + 5 \cdot C_1 + 8 \cdot C_2 + \cdots + (3n+2)C_n = (3n+4)2^{n-1}$.

ii. $C_0 + 3 \cdot C_1 + 5 \cdot C_2 + \cdots + (2n+1)C_n = (2n+2)2^{n-1}$.

iii. $2 \cdot C_0 + 7 \cdot C_1 + 12 \cdot C_2 + \cdots + (5n+2)C_n = (5n+4)2^{n-1}$.

3. Show that $C_0 - 4C_1 + 7C_2 - 10C_3 + \cdots = 0$.

4. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, then find the value of $C_0 + 2C_1 + 3C_2 + 4C_3 + \cdots + (n+1)C_n$.

5. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, then find the value of $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \cdots + \frac{nC_n}{C_{n-1}}$.

6. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then find the value of $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{(n+1)}$.

7. Find the sum of the following

i. $4 \cdot C_0 + 7 \cdot C_1 + 10 \cdot C_2 + \cdots + (3n+4)C_n$

ii. $\frac{^{15}C_1}{^{15}C_0} + 2 \cdot \frac{^{15}C_1}{^{15}C_2} + 3 \cdot \frac{^{15}C_3}{^{15}C_2} + \cdots + 15 \cdot \frac{^{15}C_{15}}{^{15}C_{14}}$

iii. $C_0 \cdot C_3 + C_1 \cdot C_4 + C_2 \cdot C_5 + \cdots + C_{n-3} \cdot C_n$

iv. $2^2 \cdot C_0 + 3^2 \cdot C_1 + 4^2 \cdot C_2 + \cdots + (n+2)^2 \cdot C_n$

v. $3 \cdot C_0 + 6 \cdot C_1 + 12 \cdot C_2 + \cdots + 3 \cdot 2^n \cdot C_n$

8.5

Multinomial Coefficients

Learning objectives:

- To generalize the Binomial theorem to Multinomial theorem.
- To generalize the concept of Binomial coefficients to Multinomial coefficients.
  AND
- To practice the related problems.

We consider the following problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, n_3, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. We wish to know how many different divisions are possible.

We note that there are ${}^{n}C_{n_1}$ possible choices for the first group; for each choice of the first group there are ${}^{n-n_1}C_{n_2}$ possible choices for the second group; for each choice of the first two groups there are ${}^{n-n_1-n_2}C_{n_3}$ possible choices for the third group; and so on. Hence it follows from the generalized version of the basic counting principle that there are

$${}^{n}C_{n_1}\,{}^{n-n_1}C_{n_2}\,{}^{n-n_1-n_2}C_{n_3}\cdots{}^{n-n_1-n_2-\cdots-n_{r-1}}C_{n_r}$$

$$=\frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{(n-n_1-n_2\cdots-n_{r-1})!}{0!n_r!}$$

$$=\frac{n!}{n_1!n_2!\cdots n_r!}$$

possible divisions.

We use the following notation:

If $n_1 + n_2 \ldots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{\boldsymbol{n}}{\boldsymbol{n_1}, \boldsymbol{n_2}, \ldots, \boldsymbol{n_r}} = \frac{\boldsymbol{n!}}{\boldsymbol{n_1!n_2!\ldots n_r!}}$$

Thus $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 1:

A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the three groups are possible?

Solution:

There are $\frac{10!}{5!2!3!} = 2520$ possible divisions.

Example 2:

Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution:

There are $\frac{10!}{5!5!} = 252$ possible divisions.

Example 3:

In order to play a game of basket ball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution:

We note that this example is different from previous example because now the order of the two teams is irrelevant. That is, there is no A and B team but just a division consisting of 2 groups of 5 each. Hence the desired answer is

$$\frac{10!}{5!5!} \div 2 = \frac{252}{2} = 126$$

The following theorem, known as ***multinomial theorem***, generalizes the binomial theorem.

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r): \\ n_1 + n_2 \ldots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 \ldots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as ***multinomial coefficients.***

For the case $r = 2, n_1 = k, n_2 = n - k,$ the equation

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!} \qquad \text{-------}(1)$$

reduces to the binomial coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Furthermore, we note that the multinomial coefficient in equation (1) is identical to the expression for the number m of distinguishable permutations of n objects, n_i of which are identical and type i (for $i = 1,2, \ldots, r$ and $n_1 + n_2 \ldots + n_r = n$):

$$m = \frac{n!}{n_1!n_2!\ldots n_r!}$$

A careful look should convince us that the two expressions must be identical.

Example 4:

$$(x_1 + x_2 + x_3)^2 = \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 + \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 + \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1$$

$$= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

IP1:

Find the number of ways that 12 apples can be divided among 4 children if the youngest child receives 6 apples and each of the other 2 apples.

Solution:

We wish to find m partitions of the 12 apples into 4 cells containing $6,2,2,2$ apples respectively. So, that

$$\therefore \frac{12!}{6!2!2!2!} = \frac{12\times11\times10\times9\times8\times7}{8} = 83160$$

IP2:

Find the coefficient of x^{2009} in the expansion of $(1-x)^{2008}(1+x+x^2)^{2007}$.

Solution:

$$(1-x)^{2008}\left(1+x+x^2\right)^{2007}=(1-x)\left[(1-x)\left(1+x+x^2\right)\right]^{2007}$$
$$=(1-x)\left(1-x^3\right)^{2007}$$
$$=\left(1-x^3\right)^{2007}-x\left(1-x^3\right)^{2007}$$

All the terms in the expansion of $\left(1-x^3\right)^{2007}$ are of the form x^{3r} and all the terms in the expansion of $x\left(1-x^3\right)^{2007}$ are of the form x^{3r+1}, where as x^{2009} is of the form x^{3r+2}. Thus, the desired coefficient is 0.

IP3:

Find the coefficient of $x^3y^3z^2$ in $(2x - 3y + 5z)^8$.

Solution:

We have $(x_1+x_2+x_3)^n = \sum_{\substack{n_1,n_2,n_3\geq 0 \\ n_1+n_2+n_3=n}} \binom{n}{n_1,n_2,n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$

Taking $x_1 = 2x$, $x_2 = -3y$, $x_3 = 5y$ and $n = 8$, we get

Now,

$$(2x-3y+5z)^8 = \sum_{\substack{n_1,n_2,n_3\geq 0 \\ n_1+n_2+n_3=8}} \binom{8}{n_1,n_2,n_3} (2x)^{n_1}(-3y)^{n_2}(5z)^{n_3}$$

$\therefore$ The coefficient of $x^3y^3z^2 = \binom{8}{3,3,2}(2)^3(-3)^3(5)^2$

$$= \frac{8!}{3!3!2!}(2)^3(-3)^3(5)^2$$

$$= 560\ (2)^3\ (-3)^3(5)^2$$

IP4:

If $(1+x+x^2+x^3)^5 = \sum_{k=0}^{15} a_k x^k$, then find $\sum_{k=0}^{7} a_{2k}$.

Solution:

$(1+x+x^2+x^3)^5 = a_0 + a_1 x + a_2 x^2 + \cdots + a_{15} x^{15}$ ------(A)

Put $x = 1$ in (A), we get

$$a_0 + a_1 + a_2 + a_3 + a_4 + \cdots + a_{15} = (1+1+1+1)^5$$

$$a_0 + a_1 + a_2 + a_3 + a_4 + \cdots + a_{15} = 4^5 \qquad \text{------}(1)$$

Put $x = -1$ in (A) , we get

$$a_0 - a_1 + a_2 - a_3 + a_4 - \cdots - a_{15} = (1-1+1-1)^5$$

$$a_0 - a_1 + a_2 - a_3 + a_4 - \cdots - a_{15} = 0 \qquad \text{------}(2)$$

By adding (1) and (2), we get

$$2(a_0 + a_2 + \cdots + a_{14}) = 4^5$$

$$a_0 + a_2 + \cdots + a_{14} = \sum_{k=0}^{7} a_{2k} = \frac{4^5}{2} = 512$$

P1:

In how many ways can the 12 students in a class take 4 different tests if 3 students are to take each test?

Solution:

We have to divide (partition) 12 students for 4 different tests if 3 students are to take each test.

$$\therefore \frac{12!}{3!3!3!3!} = 3{,}69{,}600 \text{ Partitions}$$

P2:

Find the coefficient of x^{10} in the expansion of $(1+x^2-x^3)^8$.

Solution:

We rewrite the given expression as $[1+x^2(1-x)]^8$ and expand By using the binomial theorem, we have

$$[1+x^2(1-x)]^8 = {}^8C_0 + {}^8C_1\, x^2(1-x) + {}^8C_2\, x^4(1-x)^2 + {}^8C_3\, x^6(1-x)^3$$
$$+ {}^8C_4\, x^8(1-x)^4 + {}^8C_5\, x^{10}(1-x)^5 + \ldots\ldots\ldots.$$

The two terms which contain x^{10} are ${}^8C_4\, x^8(1-x)^4$ and ${}^8C_5\, x^{10}(1-x)^5$.

Thus, the coefficient of x^{10} in the given expansion

$= {}^8C_4$ [Coefficient of x^2 in the expansion of $(1-x)^4$] $+ {}^8C_5$

$$= {}^8C_4(6) + {}^8C_5 = \frac{8!}{4!\,4!}(6) + \frac{8!}{3!\,5!} = (70)(6) + 56 = 476$$

P3:

Find coefficient of x^7 in the expansion of $\left(1-2x+x^3\right)^6$.

Solution:

$$\left(1-2x+x^3\right)^6=\sum_{\substack{p,q,r\geq 0\\ p+q+r=6}}\frac{6!}{p!\,q!\,r!}(1)^p(-2x)^q\left(x^3\right)^r$$

$$=\sum_{\substack{p,q,r\geq 0\\ p+q+r=6}}\frac{6!}{p!q!r!}(-2)^q x^{q+3r} \text{ --------(1)}$$

For coefficient of x^7, we have to take $p=1, q=4, r=1$ and $p=3, q=1,\ r=2$.

Thus, coefficient of x^7 is $=\frac{6!}{1!1!4!}(-2)^4+\frac{6!}{3!1!2!}(-2)$

$$=480-120=360$$

P4:

a. If $(3+7x-9x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$,

then find the value of

i) $a_0 + a_1 + a_2 + \cdots + a_{2n}$ ii) $a_0 + a_2 + a_4 + \cdots + a_{2n}$

b. If $(1+x+x^2+x^3)^7 = b_0 + b_1x + b_2x^2 + \cdots + b_{21}x^{21}$

then find the value of

i) $b_0 + b_1 + b_2 + \cdots + b_{21}$ ii) $b_1 + b_3 + b_5 + \cdots + b_{21}$

Solution:

a. We have

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n} = (3 + 7x - 9x^2)^n \text{ ------(1)}$$

Put $x = 1$ in (1), we get

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_{2n} = 1 \qquad \text{------(2)}$$

Put $x = -1$ in (1), we get

$$a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} = (-13)^n \qquad \text{------(3)}$$

$$(2) + (3) \implies 2(a_0 + a_2 + \cdots + a_{2n}) = 1 + (-13)^n$$

$$\implies a_0 + a_2 + a_4 + \cdots + a_{2n} = \frac{1+(-13)^n}{2}$$

b. We have

$$b_0 + b_1 x + b_2 x^2 + \cdots + b_{21} x^{21} = (1 + x + x^2 + x^3)^7 \text{-----(1)}$$

Put $x = 1$ in (1), we get

$$b_0 + b_1 + b_2 + \cdots + b_{21} = 4^7 \qquad \text{------(2)}$$

Put $x = -1$ in (1), we get

$$b_0 - b_1 + b_2 - b_3 + \cdots - b_{21} = 0 \qquad \text{------(3)}$$

$$(2) - (3) \implies 2(b_1 + b_3 + b_5 + \cdots + b_{21}) = 4^7$$

$$\implies b_1 + b_3 + b_5 + \cdots + b_{21} = 2^{13}$$

1. Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3 toys and each of the others 2 toys.

2. There are 12 students in a class. Find the number m of ways that 12 students can take 3 different tests if 4 students are to take each test.

3. Find the number m of ways that 12 students can be partitioned into 3 teams so that each team contains 4 students.

4. Find the number of ways in which 15 recruits can be drafted into three different regiments, five into each.

5. Find the number of ways in which 15 recruits can be divided into three equal groups.

6. Find the Coefficient of x^8 in the expansion of $(1-2x+x^3)^6$.

7. Find the coefficient of $x_1^{\,2}\, x_3\, x_4^{\,3}\, x_5^{\,4}$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$.

8. Find the coefficient of $p^3 q^2 r$ in the expansion of $(p + q + r + s + t)^9$.

9. Find the sum of the coefficients in the $(1 + x - 3x^2)^{171}$.

10. If $(1+3x-2x^2)^{10} = a_0 + a_1x + a_2x^2 + \cdots + a_{20}x^{20}$, then show that

a. $a_0 + a_1 + a_2 + \cdots + a_{20} = 2^{10}$

b. $a_0 - a_1 + a_2 - a_3 \ldots + a_{20} = 4^{10}$

11. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then show that

a. $a_0 + a_1 + a_2 + \cdots + a_{2n} = 3^n$

b. $a_0 + a_2 + a_4 + \cdots + a_{2n} = \frac{3^n+1}{2}$

c. $a_1 + a_3 + a_5 + \cdots + a_{2n-1} = \frac{3^n-1}{2}$

d. $a_0 + a_3 + a_6 + a_9 + \cdots = 3^{n-1}$